NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from November Number.]

SCHOLION III, *in which is weighed the attempt of the Arab Nassaradin, and likewise the idea of the illustrious John Wallis upon the same affair.*

This endeavor of the Arab Nassaradin the already eulogised John Wallis has published in the Latin language with remarks added in opportune place.

However Nassaradin requires two things to be conceded to him in this affair.

The first is; that any two straight lines lying in the same plane, upon which ever-so-many other straight lines so strike, that they are always perpendicular to one indeed of these, but always cut the other at unequal angles, truly toward one part always under an acute angle, and toward the other always under an obtuse angle; that, I say, the above mentioned lines be supposed always more (as long as they do not mutually cut) to approach each other toward the side of those acute angles; and on the other hand always more to recede from one another toward the parts of the obtuse angles.

But I indeed, if nothing else impedes Nassaradin, willingly permit what he postulates; since just that, which with him remains undemonstrated can be recognized as most rigorously demonstrated by me in Cor. II. after P. III.

The other postulate of Nassaradin is the reciprocal of the first; that indeed the angle may always be acute toward those parts where the just mentioned perpendiculars are supposed to become always shorter; but obtuse toward the other parts where these perpendiculars are supposed to go out always longer. But here lurks an ambiguity.

For why (while from any one perpendicular prescribed as the first we proceed to the others) may not the angles of the consequent perpendiculars, on the same side acute, not become even greater, even to where one strikes upon a right angle, consequently upon such a perdendicular as is itself the common perpendicular to each of the aforesaid straights? And if indeed that happens, evanishes this subtle preparation of Nassaradin, by means of which ingeniously indeed, but with great labor he demonstrates the Euclidean postulate.

And yet if Nassaradin with a certain justice may determine to presume as if known 'per se' that persistence of acute angles on the same side: why can not also (I speak with Wallis) be assumed as if clear 'per se': *Two straights in the same plane converging* (upon which of course an other straight striking makes toward the same parts two angles less than two right angles, as suppose one right, and the other in whatever way acute) *finally meet, if produced?*

Nor in fact can it be objected, that this greater convergence toward one

side can always subsist within a certain determinate limit, so that indeed a certain so much of distance always intervenes between these lines on this side, even if still one approaches always more nearly to the other.

That cannot, I say, be objected ; since from this itself I will demonstrate, after P. XXV., the meeting at a finite distance of all such straights, in accordance with the Euclidean postulate.

Now I go over to the aforesaid John Wallis, who, as made a custom with so many great men, ancient as well as recent, and on the other hand from the obligation imposed on his Oxford professional chair, determined to undertake this same duty of demonstrating the oft mentioned postulate.

Now solely he assumes as if certain, what follows: namely that *to any given figure another similar of any magnitude is possible.*

And that this indeed may be presumed of any figure (although in his affair he assumes solely a rectilineal triangle) is well argued from the circle, which of course all admit can be described with any sized radius.

Further the acute man observes most cautiously it does not thwart this his presumption, that besides the equality of corresponding angles also the proportionality of all corresponding sides is required, in order that a rectilineal figure, for example a triangle, may be similar to another rectilineal triangle ; though still the definition of proportion, and forthwith of similar figures are to be taken from the fifth, and the sixth book of Euclid: *For* (says he himself) *Euclid could have put each in front of book first.*

Hereafter, this standing (which nevertheless can be denied by any one, unless it is demonstrated) the famous man carries out his intent with really beautiful and ingenious effort.

But I am unwilling to fail in anything to the charge undertaken by me.

Therefore I assume two triangles, one ABC, and the other DEF (fig. 24) mutually equiangular. I do not say wholly similar; because I do not need the proportionality of the sides about the equal angles, nay nor any determinate measure of the sides themselves. Merely therefore I do not wish triangles mutually equilateral, since then the eighth of book first would alone suffice, without any assumption.

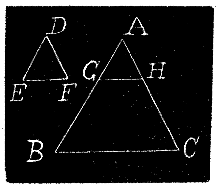

Fig. 24.

So let the angles at the points A, B, C, be equal to the angles at the points D, E, F; and let the side DE be less than the side AB; and in AB is assumed the portion AG equal to this DE, and likewise in AC the portion AH equal to this DF. But that DF must be less than AC I will make clear below. Then (GH joined) follows (from Eu. I. 4) the angles at the points E, and F will be equal to AGH, AHG. However since the just mentioned angles, together with the others BGH, CHG, are equal (Eu. I. 13) to four right angles ; likewise will be equal to four right angles the angles at the points B, and C, together with these same angles BGH, CHG. Therefore the four angles of the quadrilateral $BGHC$ will be together equal to four right angles ; and conse-

quently (from P. XVI.) is established the hypothesis of right angle; and at the same time (from P. XIII.) the Euclidean postulate.

Moreover I have supposed the side *DF*, or *AH* assumed equal to it, to be less than the side *AC*. For if it were equal, and so the point *H* should fall upon the point C, then the angle *BCA* would be equal (by hypothesis) to the angle *EFD*, or *GCA* (which then it would become) a part to the whole; which is absurd.

But if it were greater, and so the join *GH* should cut *BC* itself in some point, now the external angle *ACB* would be from the hypothesis equal (against Eu. I. 16) to the internal and opposite angle (which then would become) *AHG*, or *GHA*.

Therefore I have rightly supposed the side *DF* of one triangle to be less than the side *AC* of the other triangle, in accordance with the hypothesis now established.

Wherefore from any two triangles mutually equiangular, but not also mutually equilateral, the Euclidean postulate is established.

Quod intendebatur.

[To be Continued.]

HISTORICAL SURVEY OF THE ATTEMPTS AT THE COMPUTATION AND CONSTRUCTION OF π.

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[NOTE. The following article is translated (by permission) from Professor Klein's recent work, *Vortraege ueber ausgewaehlte Fragen der Elementargeometrie, ausgearbeitet von F. Taegert*, Leipzig, Teubner, 1895. The work can not be too highly commended to teachers, since it is one of those exceedingly rare treatises in which a master of modern mathematics has treated elementary subjects from his high point of view.
Michigan State Normal School, December, 1895.]

Later in this work it will be proved that the number π belongs to that class of numbers known as transcendent, whose existence was shown in the preceding chapter. This fact was first proved by Lindemann in 1882, and a problem was thereupon settled which, so far as our information extends, has occupied the attention of mathematicians for 4000 years, namely, that of the quadrature of the circle.

It is evident that if the number π is not algebraic it cannot be constructed by means of the compasses and ruler. Hence the quadrature of the circle is, in the sense understood by the ancients, impossible. It is of greatest interest to follow the fortunes of this problem in the various epochs of Science, as ever new attempts were made to find a solution by means of the ruler and the